A Class of Convex Polyhedra with Few Edge Unfoldings

Alex Benton* Joseph O'Rourke†

January 25, 2008

Abstract

We construct a sequence of convex polyhedra on n vertices with the property that, as $n \to \infty$, the fraction of its edge unfoldings that avoid overlap approaches 0, and so the fraction that overlap approaches 1. Nevertheless, each does have (several) nonoverlapping edge unfoldings.

1 Introduction

An *edge unfolding* of a polyhedron is a cutting of the surface along its edges that unfolds the surface to a single, nonoverlapping piece in the plane. It has long been an open question of whether or not every convex polyhedron has an edge unfolding.[1] See [DO07, Chap. 22] for background and the current status of this problem.

An early empirical investigation of this question led to the conjecture that a random edge unfolding of a random convex polyhedron of n vertices leads to overlap with probability 1 as $n \to \infty$, [SO87],[2] under any reasonable definition of "random." It is easy to see that the cuts must form a spanning tree of the polyhedron vertices. It is known that there are $2^{\Omega(\sqrt{F})}$ cut trees for a polyhedron of F faces. So the conjecture says that "most" of the exponentially many cut trees lead to overlap. Of course, even if most unfoldings overlap in this sense, this is entirely compatible with the hypothesis that there always exists at least one non-overlapping unfolding.

No progress has been made on this random-unfolding conjecture (as far as we know), but Lucier [Luc06] was able to disprove several unfolding conjectures by carefully arranged polyhedra that force what he calls *2-local overlap*.[3] Although

*DAMTP, Centre for Mathematical Science, Cambridge University, Cambridge CB3 0WA, UK. A.Benton@damtp.cam.ac.uk.

†Dept. Comput. Sci., Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu.

[1] http://cs.smith.edu/~orourke/TOPP/P9.html#Problem.9

[2] Data summarized in [DO07, p. 315].

[3] The faces that overlap are incident to the endpoints of a common edge in the unfolding.

not all our overlaps are 2-local, they are k-local (in Lucier's notation) for some small k, so our work can be viewed as following the spirit of his investigations.

In this note we construct an infinite sequence of convex polyhedra with the property that most of its unfoldings overlap, in the sense that, as $n \to \infty$, the number of its edge unfoldings that overlap approaches 1.

2 Banded Hexagon

The construction is based on a particular example from [O'R07], which showed that it is impossible to extend band unfoldings to obtain edge unfoldings of prismatoids. The details of the motivation for that work are not relevant here, but we employ its central construction, which we now describe.

Consider a hexagon formed by replacing each side of an equilateral triangle with two nearly collinear edges. The hexagon is then surrounded by a band of six identical quadrilaterals, forming a slight convexity at all edges. See Fig. 1. The six vertices of the hexagon A are $(a_0, \ldots, a_5)$, and each is connected to its

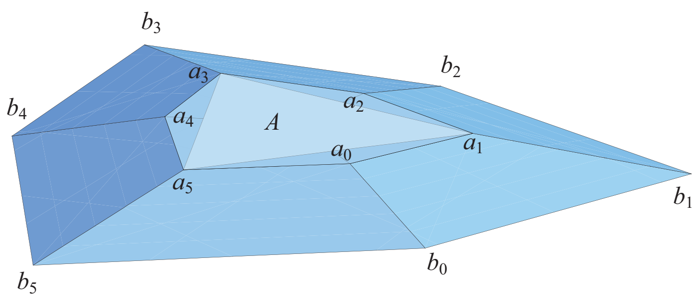

Figure 1: Banded hexagon from [O'R07].

counterpart b_i on the outer rim of the band. The slight convexity means that the curvature at the a_i vertices is small. Cutting and flattening a vertex opens it by an amount equal to the curvature.

The key property of this *banded hexagon* is as follows.

Property 1 (Hexagon Overlap) *If only one band edge $a_i b_i$ is cut (as part of the cut tree), so that the six faces of the band remain connected together, and all but one of the hexagon edges $a_i a_{i+1}$ are cut, then the unfolding overlaps.*

Fig. 2(a-c) illustrates the opening at a_3, and (d-f) the opening at a_0. The other possibilities are symmetric.

3 Banded Geodesic Domes

For the purposes of [O'R07], the band quadrilaterals were chosen to be trapezoids. However, that is not an essential property, and we modify the construction

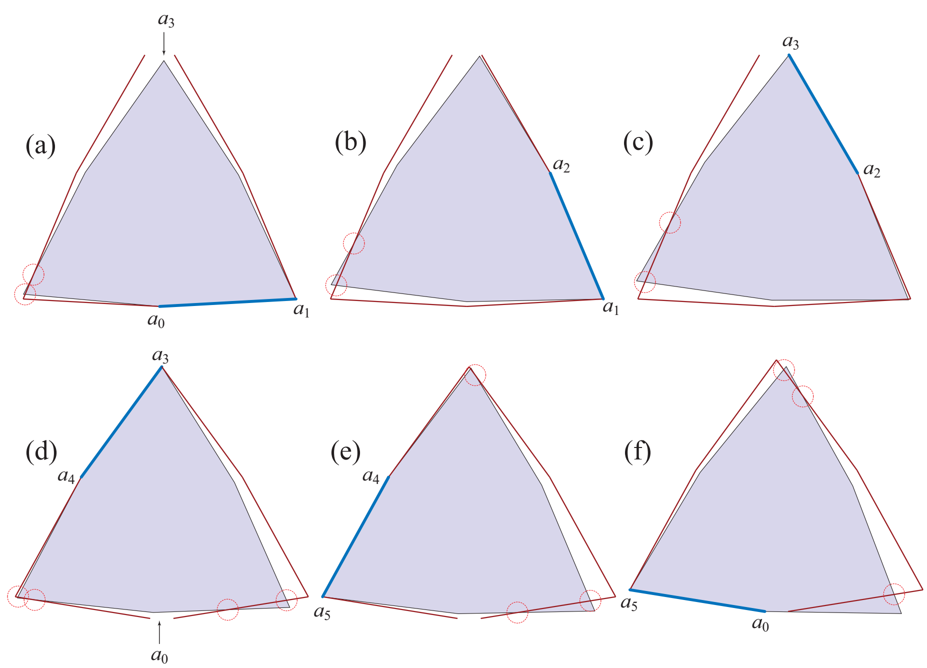

Figure 2: Placements of A when a_3 is cut (top row) and when a_0 is cut (bottom row). The attachment edge of the band to A is blue. Circles indicate overlap. The band lies outside the red rim. [Fig. 3 in [O'R07].]

here so that the quadrilaterals remain congruent but are no longer trapezoids. The Hexagon Overlap property only relies on small curvature at the a_i, and the hexagon A having three acute angles (at $\{a_1, a_3, a_5\}$) interspersed with three nearly π-angles (at $\{a_0, a_2, a_4\}$). See ahead to Fig. 8.

With this flexibility, it is possible to glue together copies of the banded hexagon construction onto a triangulated surface. We choose to use "geodesic domes" as our base polyhedron, a repeated meshing starting with the icosahedron that has nearly equilateral faces. Fig. 3 illustrates the first four *levels* of the geodesic dome construction, with each triangle face replaced by a banded hexagon. Let P_L be the banded geodesic dome refined to level L. Level $L = 0$ is based on the icosahedron. Level $L = 1$ partitions each face of the icosahedron into four equilateral triangles, and projects to the circumscribing sphere. And so on. The number of faces, edges, and vertices of the completed construction for P_L are:

$$\begin{aligned} F &= 140 \cdot 4^L \\ E &= 300 \cdot 4^L \\ n = V &= 160 \cdot 4^L \end{aligned}$$

We can drive $n \to \infty$ by choosing larger and larger values of L. At $L = 3$, there are $n = 10242$ vertices.

4 Unfoldings

Although the point of this note is that these banded geodesic domes are in some sense difficult to edge-unfold, in fact each of the four shown in Fig. 3 can unfold without overlap. Figs. 4-7 show unfoldings found by a yet-to-be-thwarted unfolding procedure described in [Ben08]. Although we have not attempted to formally prove it, it seems likely that banded geodesic domes for any L can be edge-unfolded similarly, roughly by following the geodesics.

All of these unfoldings have the property that each hexagon has two or more band cuts incident to its vertices (although these cuts are below the resolution of all but Fig. 4). We see how this avoids the Hexagon Overlap property in the next section.

5 Proof

Overview. The proof has the following overall structure. First we establish that at least a positive fraction $\rho > 0$ of all cut trees that span a finite-sized connected region C of the surface of P_L satisfy the Hexagon Overlap property, and so force unfolding overlap. Thus, at most $(1-\rho)$ of those trees avoid overlap. Then a cut tree that avoids overlap everywhere in the unfolding must avoid local overlap in each of these regions. Because the regions are a finite-size, as $L \to \infty$,

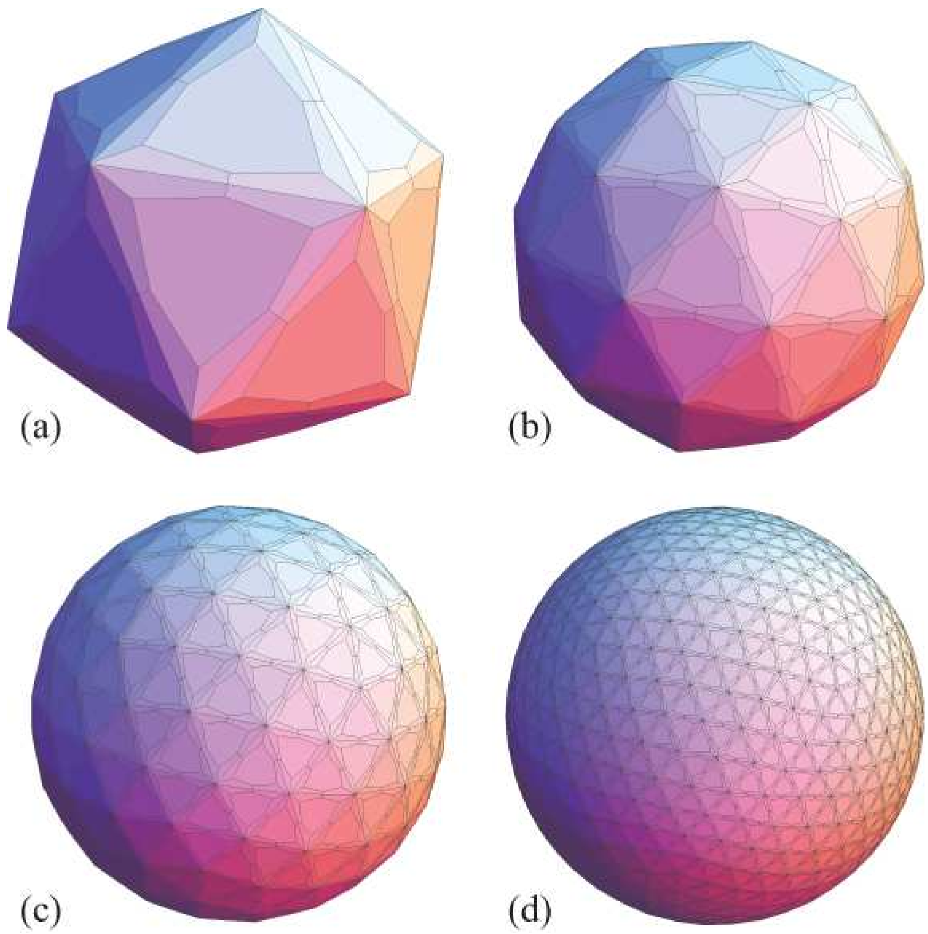

Figure 3: Banded geodesic domes for levels $L = 0, 1, 2, 3$. [Quality of this figure reduced to satisfy arXiv restrictions.]

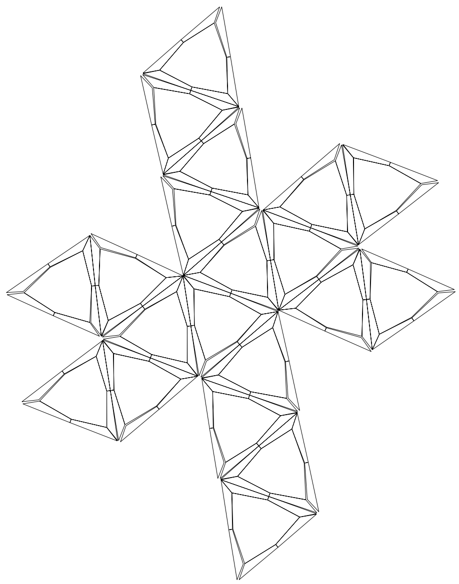

Figure 4: Edge unfolding of banded geodesic dome, $L = 0$.

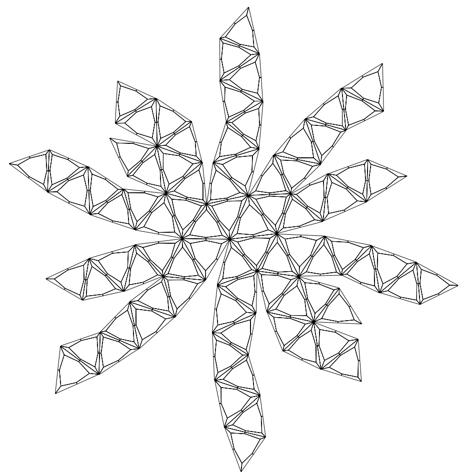

Figure 5: Edge unfolding of banded geodesic dome, $L = 1$.

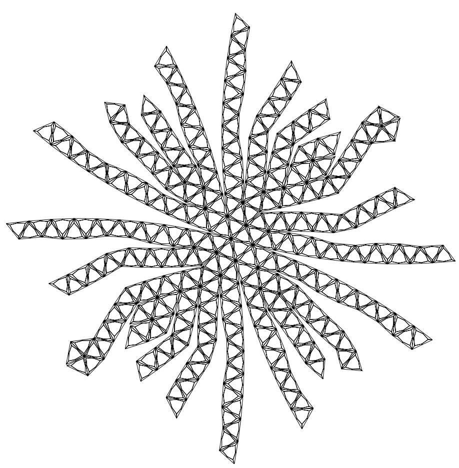

Figure 6: Edge unfolding of banded geodesic dome, $L = 2$.

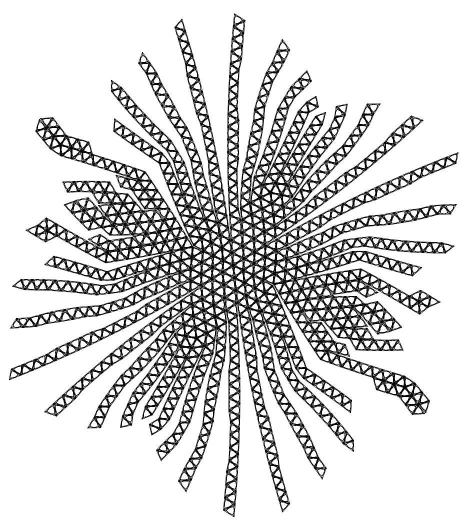

Figure 7: Edge unfolding of banded geodesic dome, $L = 3$.

the number k of regions also gets arbitrarily large. Thus the fraction of trees that avoid overlap everywhere is at most $(1-\rho)^k$, which goes to 0 as $k \rightarrow \infty$.

Connection Tree. The cut tree T is a spanning tree of the polyhedron vertices. The dual *connection* tree $T^{\triangle}$ is a spanning tree of the faces. In $T^{\triangle}$, two face nodes are connected if the faces share an uncut edge. T and $T^{\triangle}$ each uniquely determine the other. In this section we reason mostly with $T^{\triangle}$.

One Hexagon. Focus on one hexagon A of the polyhedron P. Referring to Fig. 8, let $e_i = a_i a_{i+1}$, and $u_i = a_i b_i$. The conditions that lead to Hexagon Overlap are: exactly one e_i is not cut, and exactly one u_i is cut. In terms of the dual tree $T^{\triangle}$, this means that the hexagon is a leaf node, surrounded by a band path of length 5, as in the figure. Clearly there are 6^2 such dual tree patterns leading to Hexagon Overlap (6 choices for e_i and 6 for u_j), when one banded hexagon is considered in isolation.

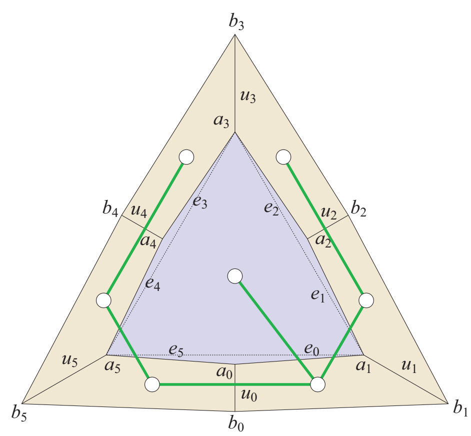

Figure 8: e_0 is not cut and u_3 is cut. All other e_i are cut and all other u_j are not cut. Dual tree $T^{\triangle}$ is shown.

Tiling Clusters. Now we consider a group of 16 banded hexagons, which together form a nearly equilateral triangular cluster, as shown in Fig. 9. Let h be the central banded hexagon in a cluster C. The choice of the size and shape of C is somewhat arbitrary. Our specific choice is motivated by two concerns: (1) The surface of P_L is nearly an equilateral lattice tiling of banded hexagons, and so can itself be tiled by copies of the nearly equilateral C, for appropriate L. (2) The central h is sufficiently "buffered" from the boundary of C, in this case by the 15 other banded hexagons of C, for a counting argument to go through. Both of these points will be revisited below.

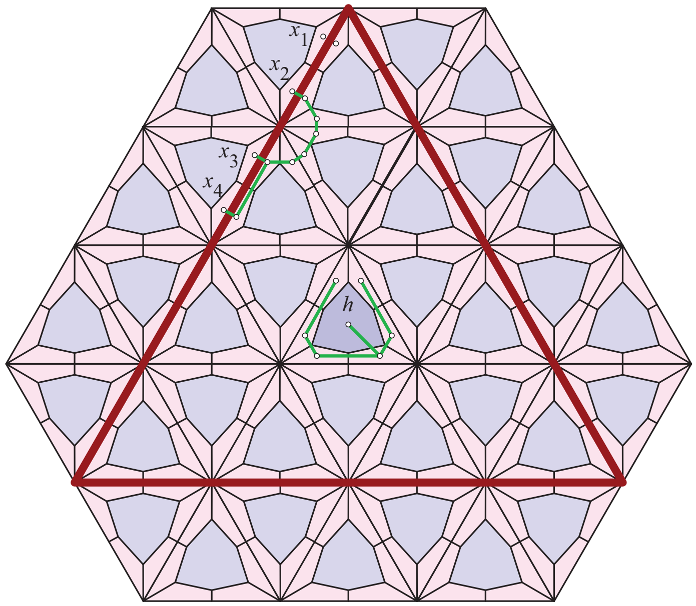

Figure 9: C: 16 banded hexagons, with central h. $x_1, \ldots, x_{24}$: surrounding quadrilateral nodes.

Counting Overlapping Trees. We now argue that there are at least a positive fraction $\rho > 0$ of trees spanning C that induce local overlap.

Let $T^{\triangle}$ be a dual spanning tree of P, and denote by $G^{\triangle}$ the forest with all nodes in C deleted. There are in general many ways to complete $G^{\triangle}$ to be a spanning tree of P. The exact number of completions is difficult to count because it depends on the structure of $G^{\triangle}$. However, we can easily obtain a crude upper bound as follows. Let E_C be the number of dual edges in C; an explicit count shows that $E_C = 228$. Any completion must either use or not use each dual edge in C. Of course many of these "bit patterns" will not complete $G^{\triangle}$ to a tree, or not to a spanning tree. But every valid completion corresponds to one of these bit patterns. Therefore, the total number of completions m satisfies $m \leq 2^{E_C}$.

Let o be the number of completions of $G^{\triangle}$ that lead to unfolding overlap. Again it would be difficult to count o exactly, but we know that the 36 patterns leading to Hexagon Overlap in h must be avoided, for each forces local overlap. Moreover, because of the buffer around h in C, all of these 36 patterns are part of some valid completion, regardless of the structure of $T^{\triangle}$ outside C. We justify this last claim below, but for now proceed with the argument, assuming $o \geq 36$.

Let $\rho = o/m$ be the fraction of completions of $G^{\triangle}$ that lead to overlap. We have a lower bound on o and an upper bound on m, so together they provide a lower bound on the ratio ρ:

$$\rho \geq 36/2^{228} \approx 10^{-67} \ .$$

The exact value of this fraction ρ is not relevant to the argument; we only need that $\rho > 0$ so that $1 - \rho < 1$.

Buffering. We return to the claim that h is sufficiently buffered within C so that for each tree that spans C, there are at least the 36 overlapping variants identified above. First we explain why the more natural choice of $C = h$ does not suffice. Suppose the forest $G^{\triangle}$ has the structure that choosing an edge dual to u_i within h creates a cycle. Then it is not a option to select this edge to complete $G^{\triangle}$ to a tree. If this occurs for two or more of the u_i, then the Hexagon Overlap pattern of Fig. 8 cannot occur within h. Thus, the structure of $G^{\triangle}$ outside C forces avoidance of the Hexagon Overlap property inside C. Thus, not every C contains something to be avoided, so to speak. We now show that our choice for C provides sufficient buffering.

Let $x_1, x_2, \ldots, x_{24}$ be the 24 quadrilateral nodes surrounding and just outside C, each with a dual edge that crosses into C. Each can be viewed as the root of a tree in the forest $G^{\triangle}$. We now show that the 36 critical patterns are part of some completion of $G^{\triangle}$ to a tree that spans C and therefore all of P_L. We first connect up all these trees in the forest into one tree via connections through the quadrilaterals incident to the border of C. One way to do this is to proceed sequentially from x_1 to x_{23}, connecting x_i to x_{i+1} if their two subtrees are not yet connected, but not making the connection if they already are part of the growing connected component. (For example, in Fig. 9, perhaps x_1 does not need to be connected to x_2, but $\{x_2, x_3, x_4\}$ should receive connections.) This connects all of $G^{\triangle}$ into a single tree without employing any of the nodes of the central h. For each of the 36 overlap patterns for h, we are free to connect up the remainder of C into a spanning tree structure, which clearly can be done in many ways. Therefore, for any tree that spans P_L and C, there are at least 36 variants inside C that overlap, and so $o \geq 36$.

We should remark that C is larger than is needed for this argument to go through (e.g., the three banded hexagons at the three corners of C are not needed), but it is easier to tile the surface if we agglomerate into a nearly equilateral C.

Tiling P_L with Clusters C. Within each original icosahedral face, the geodesic dome partitioning creates a equilateral triangle tiling, which is projected to the circumscribing sphere. Each level increases the number of triangles by a factor of 4, so every two-level increase multiplies by 16. Thus, for even $L > 0$, we can tile each original icosahedral face with our 16-hexagon clusters. This first applies for $L = 2$, Fig. 3(c).

Global Argument. Let $H = 20 \cdot 4^L$ be the number of hexagons in the polyhedron P. We showed above that at most $1 - \rho$ of the dual cut tree patterns inside a given cluster avoid overlap there (for if we fall into the ρ fraction, overlap is forced).

Imagine now constructing a complete tree $T^{\triangle}$ cluster-by-cluster in the tiling, by choosing all the nodes and arcs in $T^{\triangle}$ that span one cluster C, before moving to the next cluster. This is would be an odd way to build the tree, but with appropriate foresight, any tree could be constructed in this manner. Selecting the subforest to span a particular C leads us into the analysis of above: no matter what the structure of $G^{\triangle}$ already fixed outside of C, there is a fraction ρ of subforests that must be avoided inside C.

In order to avoid overlap in the complete unfolding, one of these overlap-avoiding patterns must be selected for each of the $\lfloor H/16 \rfloor$ clusters that tile the surface. (We use the floor function here, but as noted above, choosing L to be even makes the tiling exact.) Thus, the fraction of trees that avoid overlap within all clusters simultaneously is at most $(1-\rho)^{\lfloor H/16 \rfloor}$.

Finally, as $L \to \infty$, $H \to \infty$, and the overlap-avoiding fraction of all unfoldings goes to 0, while the overlap fraction goes to 1. This is the main claim of this note.

6 Empirical Data

The argument above only establishes a (very) loose upper bound on the ratio of the overlap-avoiding unfoldings to the total number of unfoldings. Overlap can occur for other reasons, for example, by interactions between non-adjacent faces of the polyhedron. Our computation is only concerned with avoiding a particular type of local overlap. And the argument is very cautious; for example, ρ is certainly much larger than our minuscule lower bound.

The looseness of the argument is dramatically revealed by empirical results. Because the number of cut trees is so large, it is difficult to obtain exact counts. Instead we generated random spanning cut trees, and checked each for overlap in the resulting planar unfolding.[4] For the $L = 0$ banded geodesic dome, our bound[5] says that the overlapping-avoiding fraction is at most $1 - 10^{-67}$, i.e., a random cut tree could almost always avoid overlap according to our bound. However, our simulations found only 11 non-overlapping unfoldings out of 5.5 million random cut trees, for a ratio of about 2×10^{-6}, i.e., overlap is almost never avoided, with 99.9998% of unfoldings overlapping. For higher values of L, no random cut tree led to non-overlap. Even for a level-0 banded tetrahedron, only about 9% of random unfoldings were non-overlapping. So the overlap-avoiding fraction of all unfoldings of banded geodesic domes goes to zero much, much faster than our crude analysis indicates. Correspondingly, almost all unfoldings of these domes overlap.

Some understanding of this high frequency of overlap is provided by the empirical observation that, in our random unfoldings, about 70% unfolded the

[4] The software is described in [Ben08].

[5] Technically, the bound only applies to even $L > 0$, but we use it here just for a magnitude a comparison.

seven faces of a banded hexagon connected together as a unit. This fraction is stable and apparently independent of L (and therefore of n).[6] And when a banded hexagon is unfolded as a unit, the empirically observed frequency of local overlap is about 50%. Thus, we would expect the fraction

$$1 - (1 - 0.7 \cdot 0.5)^H$$

of all unfoldings to overlap. For $L = 0$, $H = 20$, this formula (using more accurate frequencies) evaluates to 99.97%. This suggests that local overlap (within one banded hexagon unit) accounts for the majority of overlaps, for counting all overlaps only increases the frequency to 99.9998%.

7 Discussion

The process of replacing each triangular face of a polyhedron by a banded hexagon could be carried out on any triangulated polyhedron, even if the faces are not nearly equilateral as in our geodesic domes. We believe this always produces a polyhedron difficult to unfold in the sense we have established here. It may be that some "base" polyhedra will yield improvements over the geodesic domes. This remains to be explored.

Finally, through an independent argument that we will not detail, we claim that at least $3/2^{17}$ of all edge unfoldings of geodesic domes overlap, i.e., there is a fixed fraction independent of n that overlap. Our experiments reported above indicate this is a significant underestimate of the true fraction that overlap, but it is fraction that can be proved. Although this claim is in some sense weaker and less interesting than the $n \to \infty$ result, it naturally raises the question of finding (infinite) classes of polyhedra for which a larger fraction of all edge unfoldings can be proved to overlap, i.e., classes more difficult to edge-unfold.

Acknowledgments. The second author thanks Erik Demaine for several enlightening discussions on the topic of this paper.

References

[Ben08] Alex Benton. *Forthcoming.* PhD thesis, Cambridge University, 2008.

[DO07] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, July 2007. `http://www.gfalop.org`.

[Luc06] Brendan Lucier. Local overlaps in special unfoldings of convex polyhedra. In *Proc. 18th Canad. Conf. Comput. Geom.*, pages 97–100, 2006.

[6] We have not attempted a theoretical explanation for this data.

[O'R07] Joseph O'Rourke. Band unfoldings and prismatoids: A counterexample. Technical Report 087, Smith College, October 2007. arXiv:0710.0811v2 [cs.CG]; `http://arxiv.org/abs/0710.0811`.

[SO87] Catherine Schevon and Joseph O'Rourke. A conjecture on random unfoldings. Technical Report JHU-87/20, Johns Hopkins Univ., Baltimore, MD, July 1987.